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Galois Reconstruction of Finite Quantum Groups

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Let $\mathscr{C}$ be a (small) category and let $F: \mathscr{C} \to \mathscr{M}alg_f$ be a functor, where $\mathscr{M}alg_f$ is the category of finite-dimensional measured algebras over a field k (or Frobenius algebras). We construct a universal Hopf algebra $A_{aut}(F)$ such that F factorizes through a functor $\overline{F}: \mathscr{C} \to \mathscr{M}coalg_f(A_{aut}(F))$, where $\mathscr{M}coalg_f(A_{aut}(F))$ is the category of finite-dimensional measured $A_{aut}(F)$-comodule algebras. This general reconstruction result allows us to recapture a finite-dimensional Hopf algebra A from the category $\mathscr{M}coalg_f(A)$ and the forgetful functor $\omega: \mathscr{M}coalg_f(A) \to \mathscr{M}alg_f$: we have $A \cong A_{aut}(\omega)$. Our universal construction is also done in a C^*-algebra framework, and we get compact quantum groups in the sense of Woronowicz.

Key Words: quantum groups; Galois theory; Tannaka duality; groups acting on sets.

1. INTRODUCTION

We begin with the following two observations:

(1) Let $\mathscr{C}$ be a (small) category and let $F: \mathscr{C} \to Set_f$ be a functor to finite sets. Let $Aut(F)$ be the automorphism group of the functor F. Then F factorizes through a functor $\overline{F}: \mathscr{C} \to Aut(F) - Set_f$ (the category of

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finite sets on which $Aut(F)$ operates) followed by the forgetful functor,

$$\begin{tikzcd} F : \mathscr{C} \arrow[rr] \arrow[dr, dashed, "\bar{F}"'] & & Set_f \\ & Aut(F) - Set_f \arrow[ur] & \end{tikzcd}$$

(2) Let $\mathscr{C} = G - Set_f$ for a finite group G and let $\omega = F$ be the forgetful functor. Then the groups $Aut(\omega)$ and G are isomorphic.

These results are easy to prove, but are of fundamental interest. Indeed they provide the basis for Grothendieck's Galois theory [4], i.e., the axiomatic characterization of the categories of the form $G - Set_f$ (G finite or profinite group).

The aim of this paper is to describe quantum analogues of the results (1) and (2). We adopt the classical philosophy for quantum spaces and quantum groups: a quantum space is thought of as an algebra (playing the role of the function algebra) and a quantum group is thought of as a Hopf algebra. An action of a quantum group on a quantum space is a right comodule algebra structure on the underlying algebra.

Let us note that we have analogues of (1) and (2) in the framework of linear representations of algebraic groups, and we also have an axiomatic theory known as tannakian duality. This theory was developed in N. Saavedra Rivano's thesis [9] under the direction of Grothendieck. The quantum analogue (i.e., for non-commutative Hopf algebras) of tannakian duality was proved ten years ago by K. H. Ulbrich ([11], see also the texts [5, 10]).

We use non-commutative tannakian duality to prove our results. We have also been inspired by S. Wang's discovery [12] of the non-existence of the quantum automorphism group of a finite-dimensional C^*-algebra. As in [12], this led us to consider measured quantum spaces (measured algebras or Frobenius algebras, see Section 2) rather than quantum spaces. This framework is in fact very natural, especially when we have a view towards the example of comodule algebras over a finite-dimensional Hopf algebra, thanks to the existence of the Haar measure (see [6]). As a special case of our construction, we get algebraic analogues of the compact quantum groups constructed in [3, 12].

In Section 2 we fix some notations and give some preliminary results. In Section 3 we describe the construction of a universal Hopf algebra $A_{aut}(F)$ associated to any functor F: $\mathscr{C} \to \mathscr{M}alg_f$ (the category of finite-dimensional measured algebras over a field k), such that F factorizes through a functor $\bar{F}$: $\mathscr{C} \to \mathscr{M}coalg_f(A_{aut}(F))$, where $\mathscr{M}coalg_f(A_{aut}(F))$ is the category of finite-dimensional measured $A_{aut}(F)$-comodule algebras. Let us emphasize that absolutely no structure is required on the category $\mathscr{C}$. This procedure allows us to reconstruct a finite-dimensional Hopf algebra A from the category $\mathscr{M}coalg_f(A)$ and the forgetful functor ω: $\mathscr{M}coalg_f(A) \to \mathscr{M}alg_f$. Indeed we prove that the Hopf algebras A and $A_{aut}(\omega)$ are isomorphic (under a slight assumption on the base field). This result is a quantum version of (2) and is an application of the fundamental theorem of Hopf modules [6, 8]. It is also related to the duality between Hopf algebras and multiplicative unitaries [1].

In Section 4 we (briefly) describe analogues of the results of Section 3 in a C^*-algebra framework. We have to use Woronowicz' Tannaka–Krein duality [14], and we get compact quantum groups [13, 15].

2. PRELIMINARIES

The aim of this section is to fix some notations and to provide the definitions and elementary lemmas (most of them probably well known) needed in the paper. We work over a fixed field k.

Let $A = (A, m, u, \Delta, \varepsilon, S)$ be a Hopf algebra. The multiplication will be denoted by m, u: $\mathbb{C} \to A$ is the unit of A, while Δ, ε, and S are respectively the comultiplication, the counit, and the antipode of A.

The category of right A-comodules will be denoted $\mathrm{Co}_f(A)$. If V is a right A-comodule, the coaction will be denoted by α_V: $V \to V \otimes A$. The Hopf algebra A is always a right A-comodule (let us say the regular one) via the comultiplication Δ.

Let V be a right A-comodule. We denote by V_0 the associated trivial comodule: the underlying vector space is V and the coaction is $1_V \otimes u$. In this way the coaction becomes a comodule map α_V: $V \to V_0 \otimes A$.

The first two lemmas are used in the proof of Theorem 3.5. The first one is a well-known trick in Hopf algebra theory (see [8, Lemma 3.1.4]).

LEMMA 2.1. *Let A be a Hopf algebra and let V be a right A-comodule. Then the map κ_V: $V \otimes A \to V_0 \otimes A$ defined by the composition*

$$V \otimes A \xrightarrow{\alpha_V \otimes 1_A} V_0 \otimes A \otimes A \xrightarrow{1_{V_0} \otimes m} V_0 \otimes A$$

is an isomorphism of A-comodules.

Indeed, the inverse is given by $(1_V \otimes m) \circ (1_V \otimes S \otimes 1_A) \circ (\alpha_V \otimes 1_A)$. When $V = A$ we have a pentagonal operator as in [1].

Let us recall that an A-Hopf module is a vector space V which both a right A-comodule and a left A-module such that the action μ: $A \otimes V \to V$ is an A-comodule map.

LEMMA 2.2. *Let H and A be Hopf algebras and let $\pi\colon H \to A$ be a Hopf algebra morphism. Let us assume*:

(i) *There is a H-comodule algebra structure $\alpha\colon A \to A \otimes H$ on A such that $(1_A \otimes \pi)\circ \alpha = \Delta_A$.*

(ii) *The map $\kappa = \kappa_A = (1_A \otimes m_A)\circ(\Delta_A \otimes 1_A)\colon\ A \otimes A \to A_0 \otimes A$ (where A_0 denotes the trivial H-comodule structure on A) is a H-comodule map.*

Then A, endowed with its H-comodule structure and its left H-module structure induced by π, is a H-Hopf module. If furthermore the antipode of H is bijective and A is finite-dimensional then π is an isomorphism.

Proof. Let $\mu = m_A \circ(\pi \otimes 1_A)\colon H \otimes A \to A$. We must show that μ is a map of H-comodules, that is,

$$\alpha \circ \mu = (\mu \otimes 1_H)\circ(1_H \otimes 1_A \otimes m_H)\circ(1_H \otimes C_{H,A} \otimes 1_H)\circ(\Delta_H \otimes \alpha), \tag{$\star$}$$

where $C_{H,A}$ denotes the symmetry. By assumption we have

$$\begin{aligned}(1_A \otimes \alpha)&\circ(1_A \otimes m_A)\circ(\Delta_A \otimes 1_A)\\ &= (1_A \otimes m_A \otimes 1_H)\circ(\Delta_A \otimes 1_A \otimes 1_H)\circ(1_A \otimes 1_A \otimes m_H)\\ &\quad\circ(1_A \otimes C_{H,A} \otimes 1_H)\circ(\alpha \otimes \alpha)\end{aligned}$$

and since $\Delta_A = (1_A \otimes \pi)\circ \alpha$, we have $(1_A \otimes f)\circ(\alpha \otimes 1_A) = (1_A \otimes g)\circ(\alpha \otimes 1_A)$ where f and g denote the left and right side of Eq. ($\star$), respectively. Composing with $\varepsilon \otimes 1_A \otimes 1_H$, we get $f = g$ and thus A is a H-Hopf module. If H has a bijective antipode we can apply the fundamental theorem of Hopf modules [6, 8]: $A \cong A^{\mathrm{co}\,H} \otimes H$ where $A^{\mathrm{co}\,H}$ denotes the invariants of A under the coaction of H. We have $A^{\mathrm{co}\,H} \cong k$ since $A^{\mathrm{co}\,A} \cong k$ and hence $A \cong H$. The assumption (i) implies that π is surjective and if A is finite-dimensional, it is also bijective. ∎

We now come to measured algebras:

DEFINITION 2.3. Let Z be an algebra. A measure on Z is a linear form $\phi\colon Z \to k$ such that the following condition holds.

Let $a \in A$, if $\phi(ab) = 0\ \forall b \in Z$, then $a = 0$.

A measured algebra is a pair (Z, ϕ) where Z is an algebra and ϕ is a measure on Z. The category of measured algebras (denoted $\mathscr{M}alg$) has all algebra morphisms as arrows. The category of finite-dimensional measured algebras is denoted by $\mathscr{M}alg_f$.

Remark 2.4. (1) Following the classical philosophy for quantum groups and quantum spaces, a measure on an algebra is thought of as an integration of functions on the underlying quantum space.

(2) Let ϕ be a linear form on a finite-dimensional algebra Z and let $a \in Z$. We denote by $\phi(a-)$ the linear form on Z defined by $\phi(a-)(b) = \phi(ab)$. Then ϕ is a measure if and only if the map $A \to A^*$, $a \mapsto \phi(a-)$ is an isomorphism. This isomorphism is a right A-module isomorphism and hence a measured algebra is a Frobenius algebra. Conversely a Frobenius algebra always has a measure: the categories $\mathscr{M}alg_f$ and $\mathscr{F}r$ (Frobenius algebras) are equivalent.

(3) Let (Z, ϕ) be a measured algebra. If $a \in Z$ is invertible, then $\phi(a-)$ is a measure on Z.

The next Lemma 2.6 will be useful. Before we need:

LEMMA 2.5. *Let Z be a finite-dimensional algebra. We assume that the base field k contains at least n =* dim *Z non-zero distinct elements. Then every element of Z is a linear combination of invertible elements.*

Proof. We first consider the matrix algebra $M_n(k)$. If $a \in M_n(k)$ is not invertible, there exists $\lambda \in k^*$ such that $a - \lambda\mathrm{id}$ is invertible (by the assumption on the field). Hence $a = (a - \lambda\mathrm{id}) + \lambda\mathrm{id}$ is a linear combination of invertible elements. Now let us consider the regular representation L: $Z \to End(Z)$ $(End(Z) \cong M_n(k))$: $L(a)(b) = ab$. If $L(a)$ is not invertible there exists $\lambda \in k^*$ such that $L(a) - \lambda\mathrm{id} = L(a - \lambda 1)$ is invertible. But $L(a - \lambda 1)$ is a right Z-module map, and hence its inverse is also a right Z-module map, i.e., is of the form $L(b)$ for some $b \in Z$. Thus $a = (a - \lambda 1) + \lambda 1$ is a linear combination of invertible elements. ∎

C. Cibils showed me examples of algebras where the conclusion of the lemma is false if the base field does not have enough elements. In our setting the result over an arbitrary field would be sufficient for Hopf algebras.

LEMMA 2.6. *Let (Z, ϕ) be a finite-dimensional measured algebra. Let us assume that the base field satisfies the assumption of Lemma* 2.5. *Every linear form on Z is a linear combination of measures.*

Proof. Let $f \in Z^*$. Then by the assumption $f = \phi(a-)$ for some $a \in Z$. By Lemma 2.5, a can be written as a linear combination of invertible elements, and the result follows from Remark 2.4(3). ∎

DEFINITION 2.7. Let A be a Hopf algebra. A *measured* (*right*) *A-comodule algebra* is a pair (Z, ϕ) where Z is a right A-comodule algebra and ϕ is a right A-colinear measure on Z. The category of measured

A-comodule algebras (resp. finite-dimensional measured A-comodule algebras) will be denoted by $\mathcal{M}coalg(A)$ (resp. $\mathcal{M}coalg_f(A)$): the morphisms are A-colinear algebra maps.

EXAMPLE 2.8. This example is of vital importance to us. Let A be a finite-dimensional Hopf algebra. Then by [6, Theorem 1.3], there is a measure J: $A \to k$ (the Haar measure) such that (A, J) is a measured A-comodule algebra. Furthermore A is cosemisimple if and only if $J(1) \neq 0$.

3. RECONSTRUCTION RESULTS

Let $\mathscr{C}$ be a (small) category and let F: $\mathscr{C} \to \mathscr{A}lg_f$ (finite-dimensional algebras) be a functor. We are looking for an analogue of (1) in the Introduction. It seems that a version of the forthcoming Theorem 3.1 is only available for bialgebras. So we work with measured algebras, a very natural framework in view of Example 2.8.

THEOREM 3.1. *Let $\mathscr{C}$ be a (small) category and let F: $\mathscr{C} \to \mathscr{M}alg_f$ be a functor. Then there is a Hopf algebra $A_{aut}(F)$ such that F factorizes through a functor $\overline{F}$: $\mathscr{C} \to \mathscr{M}coalg_f(A_{aut}(F))$ followed by the forgetful functor,*

$$\begin{tikzcd}
F : \mathscr{C} \arrow[rr] \arrow[dr, dashed, "\overline{F}"'] & & \mathscr{M}alg_f \\
& \mathscr{M}coalg_f(A_{aut}(F)) \arrow[ur] &
\end{tikzcd}$$

The Hopf algebra $A_{aut}(F)$ has the following universal property.

If B is a Hopf algebra such that F factorizes through $\mathscr{M}coalg_f(B)$, there is a unique Hopf algebra morphism π: $A_{aut}(F) \to B$ such that the following diagram commutes,

$$\begin{tikzcd}
F \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & F \otimes A_{aut}(F) \arrow[dl, "1_F \otimes \pi"] \\
& F \otimes B &
\end{tikzcd}$$

where α and β denote the coactions of $A_{aut}(F)$ and B, respectively.

Proof. We begin by the construction of the Hopf algebra $A_{aut}(F)$. For this purpose we construct an autonomous monoidal category endowed with a fibre functor and we apply Tannaka duality.

Let $\mathscr{C}(F)$ be the following category. The objects of $\mathscr{C}(F)$ are the tensor products $F(X_1) \otimes \cdots \otimes F(X_n)$ with $X_1, \dots, X_n \in \mathrm{ob}(\mathscr{C})$ (k is an object of

$\mathscr{C}(F)$). The arrows of $\mathscr{C}(F)$ are defined to be the linear combinations of compositions of tensor products of "elementary arrows":

—arrows of the form $F(f)$, where f is an arrow in $\mathscr{C}$;

—m_X: $F(X) \otimes F(X) \to F(X)$, where $X \in \mathrm{ob}(\mathscr{C})$ and m_X is the multiplication of the algebra $F(X)$;

—u_X: $k \to F(X)$, where u_X is the unit of the algebra $F(X)$;

—ϕ_X: $F(X) \to k$ where is ϕ_X is the measure associated with the algebra $F(X)$;

—B'_X: $k \to F(X) \otimes F(X)$, where B'_X is the only arrow such that $(F(X), B_X, B'_X)$ is a left dual for $F(X)$; see [5, Sect. 9] ($B_X = \phi_X \circ m_X$ and B'_X exists since ϕ_X is a measure).

It is clear from its definition that $\mathscr{C}(F)$ is an autonomous monoidal category and we have a forgetful (fibre) monoidal functor U: $\mathscr{C}(F) \to$ $Vect_f(k)$ (finite-dimensional vector spaces). We are in position to apply Tannaka duality: let us define $A_{aut}(F) := \mathrm{End}^{\vee}(U) = coend(U)$ (see [5, 10]). Since $\mathscr{C}(F)$ is autonomous monoidal, it follows that $A_{aut}(F)$ is a Hopf algebra. We know from [5, 10] that U: $\mathscr{C}(F) \to Vect_f(k)$ factorizes through $\mathrm{Co}_f(\mathrm{End}^{\vee}(U))$. Thus every object of $\mathscr{C}(F)$ carries a natural $A_{aut}(F)$-comodule structure. Furthermore for every object X of $\mathscr{C}$, the maps m_X, u_X, and ϕ_X are $A_{aut}(F)$-comodule maps, and for every arrow f in $\mathscr{C}$, $F(f)$ is a comodule map and an algebra map: it follows that we get the desired functor $\bar{F}$: $\mathscr{C} \to \mathscr{M}coalg_f(A_{aut}(F))$.

Now let B be a Hopf algebra such that F factorizes through $\tilde{F}$: $\mathscr{C} \to \mathscr{M}coalg_f(B)$. It is easy to see that one can extend the coactions of B to all the objects in $\mathscr{C}(F)$, and then U: $\mathscr{C}(F) \to Vect_f(k)$ factorizes through $\tilde{U}$: $\mathscr{C}(F) \to \mathrm{Co}_f(B)$ (the maps B'_X are automatically comodule maps since the B_X are). The universal property of $\mathrm{End}^{\vee}(U)$ (cf. [5, 10]) gives the claimed universal property of $A_{aut}(F)$. ∎

Remark 3.2. It is clear from the proof that Theorem 3.1 also holds when $\mathscr{M}alg_f$ is the category of finite-dimensional (i.e., having a dual) measured algebras in a cocomplete braided monoidal category $\mathscr{V}$; [7, 10].

The Hopf algebra $A_{aut}(F)$ is a quantum automorphism group of the functor F in the sense of [12], and as in [12] we have:

PROPOSITION 3.3. *Let* F: $\mathscr{C} \to \mathscr{M}alg_f$ *be a functor and let* $A_{aut}(F)$ *be the Hopf algebra above. Then* $\mathrm{Hom}_{k-alg}(A_{aut}(F)) \cong Aut(F)$.

Proof. We use the notations of the proof of Theorem 3.1. We have $Aut^{\otimes}(U) \cong \mathrm{Hom}_{k-alg}(\mathrm{End}^{\vee}(U)$ [5] where $Aut^{\otimes}(U)$ is the group of automorphisms of the monoidal functor U. Therefore it is sufficient to prove that $Aut^{\otimes}(U) \cong Aut(F)$. This quite easy verification is left to the reader. ∎

EXAMPLE 3.4. (1) Let $\mathscr{C} = \{*\}$ be the category with one object and one arrow and let (Z, ϕ) be a finite-dimensional algebra. Let F be the functor defined by $F(\{*\}) = (Z, \phi)$. Let us denote by $A_{aut}(Z, \phi)$ the algebra associated by Theorem 3.1: we get algebraic versions of the Hopf C^*-algebras constructed in [12]. See [2] for the representation theory of $A_{aut}(Z, \phi)$ when Z is a semisimple algebra and ϕ is a "good" trace on Z.

(2) In [3] we constructed the (compact) quantum automorphism group of a finite graph. This construction can also be recaptured algebraically from Theorem 3.1. Let $\mathscr{G} = (V, E)$ be a finite graph where V is the set of vertices and $E \subset V \times V$ is the set of edges. Let $C(V)$ and $C(E)$ be the function algebras on V and E, respectively. Let $\phi_V\colon C(V) \to k$ (resp. $\phi_E\colon C(E) \to k$) be the classical normalized integration of functions. The quantum automorphism group of $\mathscr{G}$ can be described in the following way. Let $\mathscr{C}$ be the category with two objects $\{0, 1\}$ and two arrows $i, j\colon 0 \rightrightarrows 1$. We define a functor $F\colon \mathscr{C} \to Malg_f$ by

$$F(0) = \big(C(V), \phi_V\big), \qquad F(1) = \big(C(E), \phi_E\big),$$
$$F(i) = s_*,\ F(j) = t_*,$$

where s_* and t_* are the algebra maps induced by the source and target map, respectively. Then the algebraic version of $A_{aut}(\mathscr{G})$ defined in [3] and $A_{aut}(F)$ are isomorphic.

We now will show that Theorem 3.1 enables us to reconstruct a finite-dimensional Hopf algebra from the category of finite-dimensional measured comodule algebras and the forgetful functor. This is a quantum analogue of (2).

THEOREM 3.5. *Let A be a finite-dimensional Hopf algebra. We assume that the base field k contains at least $n =$ dim A non-zero distinct elements (cf. Lemma 2.5). Let $\mathscr{C} = \mathscr{M}coalg_f(A)$ be the category of finite-dimensional measured A-comodule algebras and let $\omega\colon \mathscr{C} \to \mathscr{M}alg_f$ be the forgetful functor. The Hopf algebras A and $A_{aut}(\omega)$ are isomorphic.*

Proof. We want to apply Lemma 2.2. For simplicity we note $H = A_{aut}(\omega)$. The universal property of H yields a Hopf algebra map $\pi\colon H \to A$ and a H-comodule structure $\alpha\colon A \to A \otimes H$ on A such that $(1_A \otimes \pi) \circ \alpha = \Delta_A$. Let A_0 be the trivial A-comodule algebra whose underlying algebra is A. We must show that A_0 is also trivial as H-comodule (i.e., $\omega(A_0) \cong \omega(A)_0$). Let ψ be a measure on A_0: then ψ is a morphism in the category $\mathscr{C}(\omega)$ of the proof of Theorem 3.1, and hence a H-comodule morphism $A_0 \to k$. By Lemma 2.6 every linear form on A_0 is a linear combination of measures, and hence a H-comodule map: this means that A_0 is trivial as H-comodule. The map $\kappa = (1_A \otimes m) \circ (\Delta_A \otimes 1_A)$: $A \otimes A \to A_0 \otimes A$ is a H-comodule map since it belongs to $\mathscr{C}(\omega)$. The antipode of H is clearly bijective since every object of the category $\mathscr{C}(\omega)$ is self-dual and hence we are in the situation of Lemma 2.2: π is an isomorphism. ∎

Remark 3.6. The conclusion of Theorem 3.5 is true whenever every linear form on A is a linear combination of measures and characters. I do not know whether this condition is always realized. More generally, under this assumption, and since Lemma 2.2 only uses arrows, Theorem 3.5 should hold for a "finite-dimensional" Hopf algebra in a cocomplete braided monoidal category with reasonable dimension theory (an epimorphism between objects of the same dimension is an isomorphism): we have a classification of Hopf modules in [7].

4. THE C^*-ALGEBRA CASE

The base field is now assumed to be the field of complex numbers. We want analogues of the results of Section 3 in a C^*-algebra framework. The ideas are essentially the same as in the previous sections, so we will be a little concise.

DEFINITION 4.1. Let Z be a C^*-algebra. A measure on Z is a positive and faithful linear form ϕ on Z: $\phi(a^*a) > 0$ for $a \neq 0$. A measured C^*-algebra is a pair (Z, ϕ) where Z is a C^*-algebra and ϕ is a measure on Z. The category of measured C^*-algebras (denoted $\mathscr{M}C^*$) has all C^*-algebras morphisms (namely $*$-homomorphisms) as arrows. The category of finite-dimensional measured C^*-algebras is denoted by $\mathscr{M}C^*_f$.

LEMMA 4.2. *Let Z be a finite-dimensional C^*-algebra: every linear form on Z can be written as a linear combination of measures.*

Proof. Let ϕ be a positive and faithful trace on Z. Every linear form on Z can be written as $\phi(a-)$ for some $a \in Z$, and the linear form $\phi(a-)$ is positive and faithful if and only if a is a positive and invertible element of Z. Any element can be written as a linear combination of positive and invertible elements, and thus we get the claimed result. ∎

We now want to start with a functor F: $\mathscr{C} \to \mathscr{M}C^*_f$ and have an analogue of Theorem 3.1: we will get compact quantum groups using Woronowicz' Tannaka–Krein duality [14].

Let us recall that a compact quantum group ([13, 15], or Woronowicz algebra or Hopf C^*-algebra with unit) is a pair (A, Δ) where A is a

C^*-algebra (with unit) and $\Delta: A \to A \otimes A$ is a coassociative $*$-homomorphism such that the sets $\Delta(A)(A \otimes 1)$ and $\Delta(A)(1 \otimes A)$ are both dense in $A \otimes A$. By abuse of notation a compact quantum group is often identified with its underlying C^*-algebra. A morphism between compact quantum groups A and B is a $*$-homomorphism $\pi: A \to B$ such that $\Delta \circ \pi = (\pi \otimes \pi) \circ \Delta$.

Given a compact quantum group A there is a canonically defined Hopf $*$-algebra A^0, which is dense in A [15]. A representation of A is a comodule of the Hopf algebra A^0.

An *action* of a compact quantum group A on a C^*-algebra Z is a unital $*$-homomorphism $\alpha: Z \to Z \otimes A$ such that there is a dense sub-$*$-algebra Z of Z for which α restricts to a coaction $\alpha: Z^0 \to Z^0 \otimes A^0$; i.e., Z^0 is a right A^0-comodule algebra. A C^*-algebra endowed with an action of A is called an A-comodule C^*-algebra.

DEFINITION 4.3. Let A be a compact quantum group. A measured A-comodule C^*-algebra is a measured C^*-algebra (Z, ϕ) such that Z is an A-comodule C^*-algebra and $\phi: Z^0 \to \mathbb{C}$ is an A^0-comodule map. The category of finite-dimensional measured A-comodule C^*-algebras will be denoted by $\mathscr{M}C_f^*(A)$: the morphisms are A-colinear $*$-homomorphisms.

THEOREM 4.4. (i) *Let $\mathscr{C}$ be a (small) category and let $F: \mathscr{C} \to \mathscr{M}C_f^*$ be a functor. Then there is a compact quantum group $A_{aut}(F)$ such that F factorizes through a functor $\overline{F}: \mathscr{C} \to \mathscr{M}C_f^*(A_{aut}(F))$ followed by the forgetful functor,*

$$\begin{tikzcd} F:\mathscr{C} \arrow[rr] \arrow[dr, dashed, "\overline{F}"'] & & \mathscr{M}C_f^* \\ & \mathscr{M}C_f^*(A_{aut}(F)) \arrow[ur] & \end{tikzcd}$$

The compact quantum group $A_{aut}(F)$ has the following universal property. If B is a compact quantum group such that F factorizes through $\mathscr{M}C_f^*(B)$, there is a unique compact quantum group morphism $\pi: A_{aut}(F) \to B$ such that the following diagram commutes,

$$\begin{tikzcd} F \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & F \otimes A_{aut}(F) \arrow[dl, "1_F \otimes \pi"] \\ & F \otimes B & \end{tikzcd}$$

where α and β denote the actions of $A_{aut}(F)$ and B, respectively.

(ii) *Let A be a finite quantum group (a finite-dimensional Hopf C^*-algebra), let $\mathscr{C} = \mathscr{M}C_f^*(A)$, and let ω be the forgetful functor. Then the compact quantum groups A and $A_{aut}(\omega)$ are isomorphic.*

Proof. The proof follows the one of Theorem 3.1. Let X be an object of $\mathscr{C}$ and let ϕ_X be the associated measure on $F(X)$. We define a scalar product on $F(X)$ by $\langle x, y \rangle = \phi_X(y^*x)$. We define a category of Hilbert spaces $\mathscr{C}'(F)$ in the following way. The objects of $\mathscr{C}'(F)$ are the Hilbert spaces $F(X_1) \otimes \cdots \otimes F(X_n)$ with $X_1, \ldots, X_n \in \mathrm{ob}(\mathscr{C})$. The "elementary arrows" of $\mathscr{C}'(F)$ are the ones of $\mathscr{C}(F)$ (defined in the proof of Theorem 3.1) plus their adjoints, and the arrows of $\mathscr{C}'(F)$ are the linear combinations of compositions and tensor product of elementary ones. In this way $\mathscr{C}'(F)$ is a concrete monoidal W^*-category with conjugates, and we can apply Theorem 1.3 of [14]. We get a compact quantum group $A_{aut}(F)$ as a "universal admissible pair." Arguing as in the proof of Theorem 3.1 and using Theorem 1.3 of [14], we get assertion (i). The assertion (ii) is proved in the same way as Theorem 3.5: we use (i) and Lemma 4.2 (which plays the role of Lemma 2.6). ∎

REFERENCES

1. S. Baaj and G. Skandalis, Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres, *Ann. Sci. École Norm. Sup.* (4) **26** (1993), 425–488.
2. T. Banica, Symmetries of a generic coaction, *Math. Ann.* **314** (1999), 763–780.
3. J. Bichon, Quantum automorphism groups of finite graphs, prépublication 22/98, Université Montpellier II, q-alg/9902029.
4. A. Grothendieck, "Revêtements étales et groupe fondamental," Lecture Notes in Math., Vol. 224, Springer-Verlag, 1971.
5. A. Joyal and R. Street, An introduction to Tannaka duality and quantum groups, *in* Lecture Notes in Math., Vol. 1488, pp. 413–492, Springer-Verlag, New York/Berlin, 1991.
6. R. G. Larson and M. Sweedler, An associative orthogonal bilinear form for Hopf algebras, *Amer. J. Math.* **91** (1969), 75–93.
7. V. V. Lyubashenko, Modular transformations for tensor categories, *J. Pure Appl. Algebra* **95**, No. 3 (1995), 279–327.
8. S. Montgomery, "Hopf Algebras and Their Action on Rings," Amer. Math. Soc., Providence, 1993.
9. N. Saavedra Rivano, "Catégories tannakiennes," Lecture Notes in Math., Vol. 265, Springer-Verlag, New York/Berlin, 1972.
10. P. Schauenburg, "Tannaka Duality for Arbitrary Hopf Algebras," Algebra Berichte, Vol. 66, Verlag Reinhard Fischer, Munich, 1992.
11. K. H. Ulbrich, On Hopf algebras and rigid monoidal categories, *Israel J. Math.* **72** (1990), 252–256.
12. S. Wang, Quantum symmetry groups of finite spaces, *Comm. Math. Phys.* **195** (1998), 195–211.
13. S. L. Woronowicz, Compact matrix pseudogroups, *Comm. Math. Phys.* **111** (1987), 613–665.
14. S. L. Woronowicz, Tannaka–Krein duality for compact matrix pseudogroups: Twisted $SU(N)$ groups, *Invent. Math.* **93** (1988), 35–76.
15. S. L. Woronowicz, Compact quantum groups, *in* "Symétries quantiques, Les Houches, 1995," pp. 845–884, North-Holland, Amsterdam, 1998.